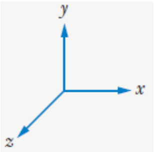
SADLER UNIT 3 MATHEMATICS SPECIALIST

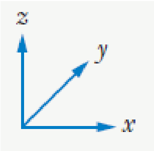
WORKED SOLUTIONS

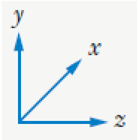
Chapter 5: Vectors in three dimensions

Exercise 5A

Question 1

a 

b 

c 

Question 2

Given $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 8\mathbf{j} - \mathbf{k}$

a $\mathbf{a} + \mathbf{b} = 5\mathbf{i} + 14\mathbf{j} + 2\mathbf{k}$

b $\mathbf{a} - \mathbf{b} = -\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

c $2\mathbf{a} + \mathbf{b} = 7\mathbf{i} + 20\mathbf{j} + 5\mathbf{k}$

d $2(\mathbf{a} + \mathbf{b}) = 10\mathbf{i} + 28\mathbf{j} + 4\mathbf{k}$

e
$$\begin{aligned}\mathbf{a} \bullet \mathbf{b} &= (2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}) \bullet (3\mathbf{i} + 8\mathbf{j} - 1\mathbf{k}) \\ &= 6 + 48 - 3 \\ &= 51\end{aligned}$$

f $\mathbf{b} \bullet \mathbf{a} = \mathbf{a} \bullet \mathbf{b} = 51$

g $|\mathbf{a}| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{49} = 7$

h $|\mathbf{a} + \mathbf{b}| = \sqrt{5^2 + 14^2 + 2^2} = \sqrt{225} = 15$

Question 3

Given $\mathbf{c} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$

a $\mathbf{c} + \mathbf{d} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$

b $\mathbf{c} - \mathbf{d} = \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$

c $2\mathbf{c} + \mathbf{d} = \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}$

d $2(\mathbf{c} + \mathbf{d}) = \begin{pmatrix} 2 \\ 8 \\ 14 \end{pmatrix}$

e $\mathbf{c} \bullet \mathbf{d} = -2 + 0 + 12 = 10$

f $\mathbf{d} \bullet \mathbf{c} = \mathbf{c} \bullet \mathbf{d} = 10$

g $|\mathbf{c}| = \sqrt{(-1)^2 + 4^2 + 3^2} = \sqrt{26}$

h $|\mathbf{c} + \mathbf{d}| = \sqrt{1^2 + 4^2 + 7^2} = \sqrt{66}$

Question 4

Given $\mathbf{e} = <1, 4, -3>$ and $\mathbf{f} = <-1,\ 2,\ 0>$

a $\mathbf{e} - \mathbf{f} = <2,\ 2,\ -3>$

b $\mathbf{e} - 2\mathbf{f} = <3,\ 0, -3>$

c $2\mathbf{e} + \mathbf{f} = <1,\ 10, -6>$

d $\mathbf{e} + \mathbf{f} = <0,\ 6, -3>$

e $\mathbf{e} \bullet \mathbf{f} = -1 + 8 + 0 = 7$

f
$$\begin{aligned}(2\mathbf{e}) \bullet (3\mathbf{f}) &= <2, 8, -3> \bullet <-3, 6, 0> \\ &= 2 \times (-3) + 8 \times 6 + (-6) \times 0 \\ &= 42\end{aligned}$$

g $(\mathbf{e} - \mathbf{f}) \bullet (\mathbf{e} - \mathbf{f}) = 4 + 4 + 9 = 17$

h
$$\begin{aligned}|\mathbf{e} - \mathbf{f}| &= |<2, 2, -3>| \\ &= \sqrt{2^2 + 2^2 + (-3)^2} \\ &= \sqrt{17}\end{aligned}$$

Question 5

Given *A*, *B* and *C* have position vectors $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $-5\mathbf{i} - 1\mathbf{j} + \mathbf{k}$ respectively.

a $\overrightarrow{AB} = (3-2)\mathbf{i} + (2-3)\mathbf{j} + [1-(-4)]\mathbf{k} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$

b $\overrightarrow{BC} = (-5-3)\mathbf{i} + (-1-2)\mathbf{j} + (1-1)\mathbf{k} = -8\mathbf{i} - 3\mathbf{j}$

c $\overrightarrow{CA} = [2-(-5)]\mathbf{i} + [3-(-1)]\mathbf{j} + (-4-1)\mathbf{k} = 7\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$

d $\overrightarrow{AC} = (-5-2)\mathbf{i} + (-1-3)\mathbf{j} + [1-(-4)]\mathbf{k} = -7\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$

[which is equal to $-\overrightarrow{\mathrm{CA}}$]

Question 6

Given $\mathbf{p}=\begin{pmatrix}3\\2\\1\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}4\\-1\\3\end{pmatrix}$, $\mathbf{r}=\begin{pmatrix}2\\0\\1\end{pmatrix}$ and $\mathbf{s}=\begin{pmatrix}4\\1\\3\end{pmatrix}$

a $\mathbf{p}+\mathbf{q}=\begin{pmatrix}7\\1\\4\end{pmatrix}$

b $\mathbf{q}+\mathbf{r}=\begin{pmatrix}6\\-1\\4\end{pmatrix}$

c $(\mathbf{p}+\mathbf{q})\bullet(\mathbf{q}+\mathbf{r})=\begin{pmatrix}7\\1\\4\end{pmatrix}\bullet\begin{pmatrix}6\\-1\\4\end{pmatrix}=42-1+16=57$

Question 7

Given $\mathbf{u}=3\mathbf{i}-2\mathbf{j}+6\mathbf{k}$ and $\mathbf{v}=2\mathbf{i}+14\mathbf{j}+5\mathbf{k}$

a $|\mathbf{u}|=\sqrt{3^2+(-2)^2+6^2}=7\,\text{units}$

b $|\mathbf{v}|=\sqrt{2^2+14^2+5^2}=15\,\text{units}$

c $\mathbf{u}\bullet\mathbf{v}=6-28+30=8$ units

d The angle between $\mathbf{u}$ and $\mathbf{v}$ is found using the fact that $\mathbf{u}\bullet\mathbf{v}=|\mathbf{u}|\times|\mathbf{v}|\times\cos\theta$.

$\cos\theta=\dfrac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{u}|\times|\mathbf{v}|}=\dfrac{8}{17\times15}$

$\theta=85.6°$ (to 1d.p.)

Question 8

Given points A and B have position vectors $\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}+2\mathbf{k}$ respectively.

$\left|\overrightarrow{OA}\right|=\sqrt{3}, \qquad \left|\overrightarrow{OB}\right|=\sqrt{9}$

$\overrightarrow{OA}\bullet\overrightarrow{OB}=2-1-2=-1$

$$\cos\theta=\frac{-1}{3\sqrt{3}}$$

$\theta=101°$ (to the nearest degree)

Question 9

Given $\mathbf{p}=-\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $\mathbf{q}=-\mathbf{i}+\mathbf{j}-2\mathbf{k}$

$|\mathbf{p}|=\sqrt{6}, \qquad |\mathbf{q}|=\sqrt{6}$

$\mathbf{p}\bullet\mathbf{q}=1$

$$\cos\theta=\frac{1}{6}$$

$\theta=80°$ (to the nearest degree)

Question 10

Given $\mathbf{s}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$ and $\mathbf{t}=\begin{pmatrix}3\\0\\3\end{pmatrix}$

$|\mathbf{s}|=\sqrt{6}, \qquad |\mathbf{t}|=3\sqrt{2}$

$\mathbf{s}\bullet\mathbf{t}=3$

$$\cos\theta=\frac{3}{6\sqrt{3}}=\frac{1}{2\sqrt{3}}$$

$\theta=73°$ (to the nearest degree)

Question 11

Given $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{s} = 3\mathbf{i} + 4\mathbf{k}$

a $|\mathbf{r}| = \sqrt{2^2 + (-3)^2 + 6^2} = 7\,\text{units}$

A unit vector in the same direction as $\mathbf{r}$ can be found by dividing $\mathbf{r}$ by $|\mathbf{r}|$.

Hence the solution is $\frac{1}{7}(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$

b $|\mathbf{s}| = \sqrt{3^2 + 0^2 + 4^2} = 5\,\text{units}$

A vector in the same direction as $\mathbf{r}$, but equal in magnitude to $\mathbf{s}$, is equal to the unit vector found in part **a** multiplied by 5.

Hence the solution is $\frac{5}{7}(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$

c $\frac{\mathbf{s}}{|\mathbf{s}|} \times |\mathbf{r}| = \frac{7}{5}(3\mathbf{i} + 4\mathbf{k})$

d $\cos\theta = \frac{\mathbf{r} \bullet \mathbf{s}}{|\mathbf{r}| \times |\mathbf{s}|} = \frac{30}{7 \times 5}$

$\theta = 31°$ (to the nearest degree)

Question 12

a $(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \times 2 = 4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$.

Hence $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is parallel to $4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$.

$(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \bullet (4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}) = 28 \neq 0$

Hence $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ is not perpendicular to $4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$.

b $(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \div 3 = \mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$, not $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

Hence $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is not parallel to $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

$(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \bullet (\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = -2 \neq 0$

Hence $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is not perpendicular to $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

c $\langle 1,3,-2\rangle \times (-2) = \langle -2,-6,4\rangle$, not $\langle -2,3,1\rangle$.

Hence $\langle 1,3,-2\rangle$ is not parallel to $\langle -2,3,1\rangle$.

$\langle 1,3,-2\rangle \bullet \langle -2,3,1\rangle = 5 \neq 0$.

Hence $\langle 1,3,-2\rangle$ is not perpendicular to $\langle -2,3,1\rangle$.

d $\langle 1,2,3\rangle \times 3 = \langle 3,6,9\rangle$, not $\langle 3,3,-3\rangle$.

Hence $\langle 1,2,3\rangle$ is not parallel to $\langle 3,3,-3\rangle$.

$\langle 1,2,3\rangle \bullet \langle 3,3,-3\rangle = 0$.

Hence $\langle 1,2,3\rangle$ is perpendicular to $\langle 3,3,-3\rangle$.

e $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \frac{5}{3} = \begin{pmatrix} 5 \\ \frac{10}{3} \\ -\frac{5}{3} \end{pmatrix} \neq \begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix}$. Hence $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix}$.

$\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix} = 15 - 14 - 1 = 0$. Hence $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix}$.

f $\begin{pmatrix} -2 \\ 6 \\ 8 \end{pmatrix} \div (-2) = \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix} \neq \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$. Hence $\begin{pmatrix} -2 \\ 6 \\ 8 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$.

$\begin{pmatrix} -2 \\ 6 \\ 8 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} = -2 - 18 + 24 = 4 \neq 0$. Hence $\begin{pmatrix} -2 \\ 6 \\ 8 \end{pmatrix}$ is not perpendicular to $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$.

g $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} \times 2 = \begin{pmatrix} 6 \\ 2 \\ 10 \end{pmatrix} \neq \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix}$. Hence $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix}$.

$\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix} = 18 + 2 - 20 = 0$. Hence $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix}$.

Question 13

Given three forces $F_1 = (5\mathbf{i} + 10\mathbf{j} + 5\mathbf{k})\,\text{N}$, $F_2 = (3\mathbf{i} + 5\mathbf{j} + 5\mathbf{k})\,\text{N}$ and $F_3 = (-2\mathbf{i} + 3\mathbf{j} - \mathbf{k})\,\text{N}$.

The resultant force is $(5\mathbf{i} + 10\mathbf{j} + 5\mathbf{k}) + (3\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}) + (-2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 6\mathbf{i} + 18\mathbf{j} + 9\mathbf{k}$.

$|6\mathbf{i} + 18\mathbf{j} + 9\mathbf{k}| = 21$ N.

Question 14

Given point A has position vector $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $\overrightarrow{BA} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

Let point B have position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$(2 - x)\mathbf{i} + (3 - y)\mathbf{j} + (-4 - z)\mathbf{k} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

$$\begin{aligned} 2 - x &= -1 \\ x &= 3 \\ 3 - y &= 3 \\ y &= 0 \\ -4 - z &= 4 \\ z &= -8 \end{aligned}$$

Hence point B has position vector $3\mathbf{i} - 8\mathbf{k}$.

Question 15

Given $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{a} - \mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$.

$$\mathbf{a} + \mathbf{b} + \mathbf{a} - \mathbf{b} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \\ -2 \end{pmatrix} = 2\mathbf{a}$$

$$\mathbf{a} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Question 16

Given $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 6 \\ p \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 7 \\ q \\ -2 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 3 \\ -4 \\ r \end{pmatrix}$.

$\mathbf{b}$ is parallel to $\mathbf{a}$ so $\mathbf{b} = \mathbf{a} \times 2$. Hence $p = 2$.

$\mathbf{c}$ is perpendicular to $\mathbf{a}$ so $14 + 3q - 2 = 0$. Hence $q = -4$.

$\mathbf{d}$ is perpendicular to $\mathbf{b}$ so $12 - 24 + 2r = 0$. Hence $r = 6$.

Question 17

a $(-4\mathbf{i} - 4\mathbf{j} + 11\mathbf{k}) + (2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = -2\mathbf{i} + 9\mathbf{k}$

The position vector of the particle after 1 second is $-2\mathbf{i} + 9\mathbf{k}$.

b $-4\mathbf{i} - 4\mathbf{j} + 11\mathbf{k} + 2(2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 4\mathbf{j} + 7\mathbf{k}$

The position vector of the particle after 2 second is $4\mathbf{j} + 7\mathbf{k}$.

c $-4\mathbf{i} - 4\mathbf{j} + 11\mathbf{k} + 3(2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 2\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$

$|2\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}| = \sqrt{2^2 + 8^2 + 5^2} = \sqrt{93}$

The particle is $\sqrt{93}$ m from O after 3 seconds.

d $\sqrt{(-4 + 2t)^2 + (-4 + 4t)^2 + (11 - 2t)^2} = 15$

$t = -\frac{2}{3}, 4.5$

So the particle will be 15 m from O after 4.5 seconds.

Question 18

$\overrightarrow{\mathrm{AB}} = -4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$

$\overrightarrow{\mathrm{BC}} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$

$\overrightarrow{\mathrm{AB}}$ is parallel to $\overrightarrow{\mathrm{BC}}$ and they meet at point B.

Therefore A, B and C are collinear.

Question 19

$$\overrightarrow{AB} = \begin{pmatrix} -5 \\ 5 \\ -5 \end{pmatrix}$$

$\begin{pmatrix} 3 + \frac{2}{5} \times (-5) \\ 4 + \frac{2}{5} \times 5 \\ 4 + \frac{2}{5} \times (-5) \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix}$ is the point that divides AB internally in the ratio 2 : 3.

Question 20

$\overrightarrow{AB} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

Let point P have position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$\overrightarrow{BP} = (x-4)\mathbf{i} + (y+1)\mathbf{j} + (z-1)\mathbf{k}$

Since $\overrightarrow{AB} = \overrightarrow{BP}$, $(x-4)\mathbf{i} + (y+1)\mathbf{j} + (z-1)\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$x = 5$
$y = -4$
$z = 3$

The position vector for point P is $5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$.

Question 21

$\overrightarrow{AB} = 4\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}$

The position vector of point P is $5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \frac{3}{4}(4\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}) = 8\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$.

Question 22

$\overrightarrow{AB} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

$\overrightarrow{BC} = 4\mathbf{i} - 3\mathbf{j} - \mathbf{k}$

$\overrightarrow{AB} \bullet \overrightarrow{BC} = -8 + 9 - 1 = 0$, so $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$.

Hence ΔABC is right angled.

Question 23

Vector $\mathbf{i}$ runs along the x-axis, $\mathbf{j}$ runs along the y-axis and $\mathbf{k}$ runs along the z-axis.

$\mathbf{a} \bullet \mathbf{i} = 2 + 0 + 0 = 2, \qquad \mathbf{a} \bullet \mathbf{j} = 3, \qquad \mathbf{a} \bullet \mathbf{k} = -1$

$|\mathbf{a}| = \sqrt{14}$

Let α be the angle that $\mathbf{a}$ makes with the x-axis, β be the angle that $\mathbf{a}$ makes with the y-axis and θ be the angle that $\mathbf{a}$ the z-axis.

$$\cos\alpha = \frac{2}{\sqrt{14}}$$

$$\alpha = 57.7°$$

$$\cos\beta = \frac{3}{\sqrt{14}}$$

$$\beta = 36.7°$$

$$\cos\theta = \frac{-1}{\sqrt{14}}$$

$\theta = 105.5°$, which is an obtuse angle so $\theta = 180 - 105.5° = 74.5°$ is the acute angle.

Question 24

For $\mathbf{d} = 7\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$:

$$\lambda + 2\mu + 4\eta = 7$$
$$-2\lambda + \mu - \eta = -5$$
$$3\lambda - \mu + 3\eta = 10$$

Solving gives $\lambda = 1$, $\mu = -1$ and $\eta = 2$.

Hence $\mathbf{d} = \mathbf{a} - \mathbf{b} + 2\mathbf{c}$.

For $\mathbf{e} = \mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$:

$$\lambda + 2\mu + 4\eta = 1$$
$$-2\lambda + \mu - \eta = -5$$
$$3\lambda - \mu + 3\eta = 8$$

Solving gives $\lambda = 1$, $\mu = -2$ and $\eta = 1$.

Hence $\mathbf{e} = \mathbf{a} - 2\mathbf{b} + \mathbf{c}$.

For $\mathbf{f} = 2\mathbf{j} - \mathbf{k}$:

$$\lambda + 2\mu + 4\eta = 0$$
$$-2\lambda + \mu - \eta = 2$$
$$3\lambda - \mu + 3\eta = -2$$

Solving gives $\lambda = -2$, $\mu = -1$ and $\eta = 1$.

Hence $\mathbf{f} = -2\mathbf{a} - \mathbf{b} + \mathbf{c}$.

Question 25

a $\overrightarrow{DC} = 10\mathbf{i}$

$\overrightarrow{DB} = 10\mathbf{i} + 4\mathbf{k}$

$\overrightarrow{DI} = 3\mathbf{j} + \mathbf{k}$

b $\overrightarrow{ID} \bullet \overrightarrow{DB} = 4$

$\left|\overrightarrow{ID}\right| = \sqrt{10}$

$\left|\overrightarrow{DB}\right| = 10$

$$\cos\theta = \frac{4}{10\sqrt{10}}$$

$\theta = 83°$ (to the nearest degree)

Question 26

a $\overrightarrow{OA} = \langle 4, 2, 0 \rangle$

$\left|\overrightarrow{OA}\right| = \sqrt{20}$

$\overrightarrow{AE} = \langle -4, -2, 8 \rangle$

$\left|\overrightarrow{AE}\right| = \sqrt{84}$

$\overrightarrow{OA} \bullet \overrightarrow{AE} = -16 - 4 = -20$

$$\cos\theta = \frac{-20}{\sqrt{20} \times \sqrt{84}}$$

$\theta = 60.8°$

b $\overrightarrow{DB} = \langle -8, 4, 0 \rangle$

$\left|\overrightarrow{DB}\right| = \sqrt{80}$

$\overrightarrow{AE} = \langle -4, -2, 8 \rangle$

$\left|\overrightarrow{AE}\right| = \sqrt{84}$

$\overrightarrow{DB} \bullet \overrightarrow{AE} = 24$

$$\cos\theta = \frac{24}{\sqrt{84} \times \sqrt{80}}$$

$\theta = 73.0°$

Question 27

a $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$

b $\left|\overrightarrow{AB}\right| = \sqrt{6^2 + 2^2 + 1^2} = \sqrt{41}$

$\left|\overrightarrow{AC}\right| = \sqrt{5^2 + 0^2 + 4^2} = \sqrt{41}$

As two sides are equal in length ΔABC is isosceles.

c $\overrightarrow{AC} \bullet \overrightarrow{AC} = 5^2 + 0^2 + 4^2 = 41$

d $\cos\angle CAB = \dfrac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \dfrac{30+0+4}{\sqrt{41}\sqrt{41}} = \dfrac{34}{41}$

giving $\angle CAB \approx 34°$

$180 - 34 = 146°$ (angle sum of triangle is 180°)

$\angle ABC = \angle ACB = \dfrac{146°}{2} = 73°$

Question 28

$\overrightarrow{GE} = \mathbf{i} - \mathbf{j} + 0\mathbf{k}$, $\left|\overrightarrow{GE}\right| = \sqrt{2}$

$\overrightarrow{OG} = 0\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\left|\overrightarrow{OG}\right| = \sqrt{2}$

$\overrightarrow{OE} = \mathbf{i} + \mathbf{j} + 0\mathbf{k}$, $\left|\overrightarrow{OE}\right| = \sqrt{2}$

$\left|\overrightarrow{GE}\right| = \left|\overrightarrow{OG}\right| = \left|\overrightarrow{OE}\right|$ so ΔOGE is an equilateral triangle.

$\overrightarrow{OB} = \mathbf{i} + \mathbf{j} + 0\mathbf{k}$, $\left|\overrightarrow{OB}\right| = \sqrt{2}$

$\overrightarrow{BE} = 0\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\left|\overrightarrow{BE}\right| = \sqrt{2}$

$\left|\overrightarrow{OB}\right| = \left|\overrightarrow{BE}\right| = \left|\overrightarrow{OE}\right|$ so ΔOBE is an equilateral triangle.

$\overrightarrow{BG} = -\mathbf{i} + \mathbf{j} + 0\mathbf{k}$, $\left|\overrightarrow{BG}\right| = \sqrt{2}$

$\left|\overrightarrow{BG}\right| = \left|\overrightarrow{BE}\right| = \left|\overrightarrow{GE}\right|$ so ΔBGE is an equilateral triangle.

$\overrightarrow{OG} = 0\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\left|\overrightarrow{OG}\right| = \sqrt{2}$

$\left|\overrightarrow{OG}\right| = \left|\overrightarrow{BO}\right| = \left|\overrightarrow{BG}\right|$ so ΔOGB is an equilateral triangle.

OBGE has four congruent equilateral triangular faces, therefore it is a regular tetrahedron.

Question 29

$\overrightarrow{OF} = \mathbf{a} + \mathbf{c} + \mathbf{d}$

Midpoint of $\overrightarrow{OF}$ occurs at $\frac{1}{2}(\mathbf{a} + \mathbf{c} + \mathbf{d})$

$\overrightarrow{AG} = -\mathbf{a} + \mathbf{c} + \mathbf{d}$

Midpoint of $\overrightarrow{AG}$ occurs at $\frac{1}{2}(-\mathbf{a} + \mathbf{c} + \mathbf{d})$

$\overrightarrow{BD} = -\mathbf{a} - \mathbf{c} + \mathbf{d}$

Midpoint of $\overrightarrow{BD}$ occurs at $\frac{1}{2}(-\mathbf{a} - \mathbf{c} + \mathbf{d})$

$\overrightarrow{CE} = \mathbf{a} - \mathbf{c} + \mathbf{d}$

Midpoint of $\overrightarrow{CE}$ occurs at $\frac{1}{2}(\mathbf{a} - \mathbf{c} + \mathbf{d})$

$\frac{1}{2}\mathbf{a} = -\frac{1}{2}\mathbf{a}$

$\frac{1}{2}\mathbf{c} = -\frac{1}{2}\mathbf{c}$

$\frac{1}{2}\mathbf{d} = -\frac{1}{2}\mathbf{d}$

Hence these midpoints are all at the same point, where the four diagonals intersect.

Question 30

Given OC is perpendicular to BA, $\mathbf{c} \bullet (-\mathbf{b} + \mathbf{a}) = 0$

$$-\mathbf{b} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{c} = 0$$
$$\mathbf{a} \bullet \mathbf{c} = \mathbf{b} \bullet \mathbf{c}$$

and OB is perpendicular to CA, $\mathbf{b} \bullet (-\mathbf{c} + \mathbf{a}) = 0$

$$-\mathbf{b} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{b} = 0$$
$$\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{c}$$

$$\begin{aligned}\overrightarrow{OA} \bullet \overrightarrow{CB} &= \mathbf{a} \bullet (-\mathbf{c} + \mathbf{b}) \\ &= -\mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{b} \\ &= -\mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{c} \\ &= 0\end{aligned}$$

Hence OA must be perpendicular to CB.

Exercise 5B

Question 1

If two vectors are parallel, one is a scalar multiple of the other.

$\mathbf{r} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$
$\mathbf{s} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$

$$\mathbf{r} \times \mathbf{s} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Due to the fact that a and b are constant, as $\mathbf{s}$ is a scalar multiple of $\mathbf{r}$.

Question 2

Given $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

Coefficients are	2	3	–1
	–1	3	1

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= [3 - (-3)]\mathbf{i} - (2 - 1)\mathbf{j} + [6 - (-3)]\mathbf{k} \\ &= 6\mathbf{i} - \mathbf{j} + 9\mathbf{k}\end{aligned}$$

$\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) = 2 \times 6 + 3 \times (-1) - 1 \times 9 = 0$, so $\mathbf{a}$ is perpendicular to $\mathbf{a} \times \mathbf{b}$.
$\mathbf{b} \bullet (\mathbf{a} \times \mathbf{b}) = -1 \times 6 + 3 \times (-1) + 1 \times 9 = 0$, so $\mathbf{b}$ is perpendicular to $\mathbf{a} \times \mathbf{b}$.

Question 3

Given $\mathbf{c} = 5\mathbf{i} + \mathbf{k}$ and $\mathbf{d} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Coefficients are	5	0	1
	1	1	1

$$\begin{aligned}\mathbf{c} \times \mathbf{d} &= (0 - 1)\mathbf{i} - (5 - 1)\mathbf{j} + (5 - 0)\mathbf{k} \\ &= -\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}\end{aligned}$$

$\mathbf{c} \bullet (\mathbf{c} \times \mathbf{d}) = 5 \times (-1) + 0 \times (-4) + 1 \times 5 = 0$, so $\mathbf{c}$ is perpendicular to $\mathbf{c} \times \mathbf{d}$.
$\mathbf{d} \bullet (\mathbf{c} \times \mathbf{d}) = 1 \times (-1) + 1 \times (-4) + 1 \times 5 = 0$, so $\mathbf{d}$ is perpendicular to $\mathbf{c} \times \mathbf{d}$.

Question 4

Given $\mathbf{p} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{q} = -\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$

Coefficients are
$$\begin{matrix} 1 & 3 & -2 \\ -1 & 6 & -4 \end{matrix}$$

$$\mathbf{p} \times \mathbf{q} = [-12 - (-12)]\mathbf{i} - (-4 - 2)\mathbf{j} + [6 - (-3)]\mathbf{k} = 6\mathbf{j} + 9\mathbf{k}$$

$\mathbf{p} \bullet (\mathbf{p} \times \mathbf{q}) = 1 \times 0 + 3 \times 6 + (-2) \times 9 = 0$, so $\mathbf{p}$ is perpendicular to $\mathbf{p} \times \mathbf{q}$.

$\mathbf{q} \bullet (\mathbf{p} \times \mathbf{q}) = -1 \times 0 + 6 \times 6 + (-4) \times 9 = 0$, so $\mathbf{q}$ is perpendicular to $\mathbf{p} \times \mathbf{q}$.

Question 5

Because $\mathbf{a}$ runs along the x-axis and $\mathbf{b}$ runs along the y-axis, you would expect the cross product (which gives the vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$) to be a vector equation running along the z-axis. Hence expect $\mathbf{a} \times \mathbf{b} = \mathbf{k}$.

Using the formula and given $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$

Coefficients are
$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$$

$$\mathbf{a} \times \mathbf{b} = (0 - 0)\mathbf{i} - (0 - 0)\mathbf{j} + (1 - 0)\mathbf{k} = \mathbf{k}$$

It follows that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

Question 6

Given $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

a Coefficients are
$$\begin{matrix} -1 & 2 & -1 \\ -3 & 2 & -1 \end{matrix}$$

$$\mathbf{a} \times \mathbf{b} = [-2 - (-2)]\mathbf{i} - (1 - 3)\mathbf{j} + [-2 - (-6)]\mathbf{k} = 2\mathbf{j} + 4\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{20} = 2\sqrt{5}$$

b $\mathbf{a} \bullet \mathbf{b} = 3 + 4 + 1 = 8$

$$|\mathbf{a}| = \sqrt{1 + 4 + 1} = \sqrt{6}$$

$$|\mathbf{b}| = \sqrt{9 + 4 + 1} = \sqrt{14}$$

$$\cos\theta = \frac{8}{\sqrt{6} \times \sqrt{14}}$$

$$\theta = 29.2^\circ$$

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta = \sqrt{6} \times \sqrt{14} \times \frac{\sqrt{105}}{21} = 2\sqrt{5}$$

Question 7

$\mathbf{p} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

Coefficients 2 –3 1

1 2 –3

$$\mathbf{p} \times \mathbf{q} = (9-2)\mathbf{i} - (-6-1)\mathbf{j} + \left[4-(-3)\right]\mathbf{k} = 7\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$$
$$\left|\mathbf{p} \times \mathbf{q}\right| = \sqrt{7^2 + 7^2 + 7^2} = 7\sqrt{3}$$

A unit vector normal to the plane containing $\mathbf{p}$ and $\mathbf{q}$ is

$$\begin{aligned}\frac{\mathbf{p} \times \mathbf{q}}{\left|\mathbf{p} \times \mathbf{q}\right|} &= \frac{7\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}}{7\sqrt{3}} \\ &= \frac{7(\mathbf{i} + \mathbf{j} + \mathbf{k})}{7\sqrt{3}} \\ &= \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \qquad \left[\text{OR } -\frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})\right]\end{aligned}$$

Question 8

$$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}, \ \overrightarrow{\mathrm{BC}} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$\begin{aligned}\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{BC}} &= (0-2)\mathbf{i} - (6-4)\mathbf{j} + (-3-0)\mathbf{k} \\ &= -2\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}\end{aligned}$$
$$\begin{aligned}\left|\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{BC}}\right| &= \sqrt{(-2)^2 + (-2)^2 + (-3)^2} \\ &= \sqrt{17}\end{aligned}$$

A unit vector perpendicular to the plane containing A, B and C is

$$\begin{aligned}\frac{\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{BC}}}{\left|\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{BC}}\right|} &= \frac{1}{\sqrt{17}}(-2\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) \\ &= -\frac{1}{\sqrt{17}}(2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \qquad \left[\text{OR } \frac{1}{\sqrt{17}}(2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\right]\end{aligned}$$

Exercise 5C

Question 1

Given a point with position vector $3\mathbf{i}+2\mathbf{j}-\mathbf{k}$, a line through the point is parallel to $2\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

a $\mathbf{r}=3\mathbf{i}+2\mathbf{j}-\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{j}+2\mathbf{k})$

b $x=3+2\lambda,\ y=2-\lambda,\ z=-1+2\lambda$

Question 2

a $\overrightarrow{\mathrm{BA}}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$

The vector equation of the line that passes through point A and point B is:

$\mathbf{r}=4\mathbf{i}+2\mathbf{j}+3\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}+2\mathbf{k})$

b $x=4+\lambda,\ y=2+\lambda,\ z=3+2\lambda$

Question 3

$(3\mathbf{i}-\mathbf{j}+5\mathbf{k})\bullet(2\mathbf{i}-3\mathbf{j}+2\mathbf{k})=6+3+10=19$

The plane that is perpendicular to the vector $3\mathbf{i}-\mathbf{j}+5\mathbf{k}$ and that contains point A has vector equation $\mathbf{r}\bullet(3\mathbf{i}-\mathbf{j}+5\mathbf{k})=19$.

Question 4

$$\begin{pmatrix}5\\1\\3\end{pmatrix}\bullet\begin{pmatrix}2\\1\\-3\end{pmatrix}=10+1-9=2$$

The plane that is perpendicular to the vector $\begin{pmatrix}5\\1\\3\end{pmatrix}$ and that contains point A has vector equation

$\mathbf{r}\bullet\begin{pmatrix}5\\1\\3\end{pmatrix}=2$.

Question 5

The plane that contains point A, $2\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ and that is parallel to the vectors $2\mathbf{i}+\mathbf{j}$ and $3\mathbf{i}-4\mathbf{j}+6\mathbf{k}$ has vector equation $\mathbf{r}=2\mathbf{i}+3\mathbf{j}-2\mathbf{k}+\lambda(2\mathbf{i}+\mathbf{j})+\mu(3\mathbf{i}-4\mathbf{j}+6\mathbf{k})$.

Question 6

The plane that contains point A, $\begin{pmatrix}-3\\2\\-1\end{pmatrix}$ and that is parallel to the vectors $\begin{pmatrix}2\\0\\-3\end{pmatrix}$ and $\begin{pmatrix}1\\-3\\2\end{pmatrix}$ has vector equation $\mathbf{r}=\begin{pmatrix}-3\\2\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\-3\end{pmatrix}+\mu\begin{pmatrix}1\\-3\\2\end{pmatrix}$.

Question 7

$a=2-3\lambda$

$7=b+\lambda$

$5=-1+2\lambda$

From the least equation $\lambda=3$.

$a=2-3\times 3=-7$

$7=b+\lambda,\ b=4$

Solution $a=-7$ and $b=4$.

Question 8

$\mathbf{r}\bullet\begin{pmatrix}3\\2\\-1\end{pmatrix}=21$

$\begin{pmatrix}x\\y\\z\end{pmatrix}\bullet\begin{pmatrix}3\\2\\-1\end{pmatrix}=21$

$3x+2y-z=21$

Question 9

Given $2x-3y+7z=5$

$\mathbf{r}\bullet(2\mathbf{i}-3\mathbf{j}+7\mathbf{k})=5$

Question 10

The plane $\mathbf{r}\bullet(3\mathbf{i}+2\mathbf{j}-\mathbf{k})=c$ has normal vector $\langle 3,\ 2,\ -1\rangle$.

The line $\mathbf{r}=2\mathbf{i}+8\mathbf{j}-3\mathbf{k}+\lambda(-6\mathbf{i}-4\mathbf{j}+2\mathbf{k})$ has vector $\langle -6,-4,2\rangle$.

$-2\times\langle 3,\ 2,\ -1\rangle=\langle -6,-4,2\rangle$

Because the line is a scalar multiple of the normal to the plane, the line is parallel to the normal to the plane. The normal is perpendicular to the plane and so is any line parallel to the normal.

Hence the line $\mathbf{r}=2\mathbf{i}+8\mathbf{j}-3\mathbf{k}+\lambda(-6\mathbf{i}-4\mathbf{j}+2\mathbf{k})$ is parallel to the plane $\mathbf{r}\bullet(3\mathbf{i}+2\mathbf{j}-\mathbf{k})=c$.

Question 11

Given $L_1:\mathbf{r}=\begin{pmatrix}10\\5\\-2\end{pmatrix}+\lambda\begin{pmatrix}4\\1\\-2\end{pmatrix}$, $\qquad L_2:\mathbf{r}=\begin{pmatrix}0\\8\\-6\end{pmatrix}+\mu\begin{pmatrix}-1\\-3\\5\end{pmatrix}$

The lines intersect when $L_1=L_2$

$$\begin{pmatrix}10\\5\\-2\end{pmatrix}+\lambda\begin{pmatrix}4\\1\\-2\end{pmatrix}=\begin{pmatrix}0\\8\\-6\end{pmatrix}+\mu\begin{pmatrix}-1\\-3\\5\end{pmatrix}$$

$$\begin{aligned}10+4\lambda&=-\mu\\5+\lambda&=8-3\mu\\-2-2\lambda&=-6+5\mu\end{aligned}$$

Solving gives $\lambda=-3$ and $\mu=2$.

The point where the lines intersect has position vector $\begin{pmatrix}10\\5\\-2\end{pmatrix}-3\begin{pmatrix}4\\1\\-2\end{pmatrix}=\begin{pmatrix}-2\\2\\4\end{pmatrix}$.

Question 12

Given

$$L_1 : \mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$$
$$L_2 : \mathbf{r} = 3\mathbf{i} + 13\mathbf{j} - 15\mathbf{k} + \mu(-\mathbf{i} + 4\mathbf{k})$$

The lines intersect when $L_1 = L_2$

$$\begin{aligned} 1 - \lambda &= 3 - \mu \\ -2 + 3\lambda &= 13 \\ 3 + 2\lambda &= -15 + 4\mu \end{aligned}$$

Solving gives $\lambda = 5$ and $\mu = 7$.

The point where the lines intersect has position vector

$$\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + 5(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) = -4\mathbf{i} + 13\mathbf{j} + 13\mathbf{k}$$

Question 13

Given

$$L_1 : \mathbf{r} = 13\mathbf{i} + \mathbf{j} + 8\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$$
$$L_2 : \mathbf{r} = 12\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} + \mu(5\mathbf{i} + 3\mathbf{j} - 8\mathbf{k})$$
$$L_3 : \mathbf{r} = -5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \beta(2\mathbf{i} + \mathbf{j} - \mathbf{k})$$

a Equate L_1 and L_2

$$13\mathbf{i} + \mathbf{j} + 8\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 12\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} + \mu(5\mathbf{i} + 3\mathbf{j} - 8\mathbf{k})$$
$$\begin{aligned} 13 + 2\lambda &= 12 + 5\mu \\ 1 - \lambda &= 2 + 3\mu \\ 8 + 3\lambda &= 6 - 8\mu \end{aligned}$$

Solving the first two equations gives $\lambda = -\dfrac{8}{11}$ and $\mu = -\dfrac{1}{11}$.

Solving the last two equations gives $\lambda = 2$ and $\mu = -1$.

As there is not a common solution, the two lines do not cross.

b Equating L_1 and L_3 gives:

$$13\mathbf{i}+\mathbf{j}+8\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{j}+3\mathbf{k})=-5\mathbf{i}+2\mathbf{j}-3\mathbf{k}+\beta(2\mathbf{i}+\mathbf{j}-\mathbf{k})$$

$$13+2\lambda=-5+5\beta$$

$$1-\lambda=2+\beta$$

$$8+3\lambda=-3-\beta$$

Solving gives $\lambda=-5$ and $\beta=4$

The point where the lines intersect has position vector $\mathbf{r}=13\mathbf{i}+\mathbf{j}+8\mathbf{k}-5(2\mathbf{i}-\mathbf{j}+3\mathbf{k})$

$\mathbf{r}=3\mathbf{i}+6\mathbf{j}-7\mathbf{k}$

$L_1 \bullet L_3=2\times2+(-1)\times1+3\times(-1)=0$

So L_1 and L_3 are perpendicular, the angle between them is 90°.

Question 14

a When $\lambda=-1$, L has vector equation

$\mathbf{r}=\mathbf{i}-2\mathbf{j}+5\mathbf{k}-1(5\mathbf{i}+3\mathbf{j}-2\mathbf{k})=-4\mathbf{i}-5\mathbf{j}+7\mathbf{k}$

This is the position vector of point A, so point A lies on line L.

b When $\lambda=1.5$, L has vector equation

$\mathbf{r}=\mathbf{i}-2\mathbf{j}+5\mathbf{k}+1.5(5\mathbf{i}+3\mathbf{j}-2\mathbf{k})=8.5\mathbf{i}+2.5\mathbf{j}+2\mathbf{k}$

The position on the z-plane is the same as point B but not any of the other positions so point B does not lie on line L.

c For point A $(-4\mathbf{i}-5\mathbf{j}+7\mathbf{k})\bullet(-\mathbf{i}+3\mathbf{j}+2\mathbf{k})=-4\times(-1)+(-5)\times3+7\times2=3$, which matches plane Π. Point A lies in the plane.

For point B $(10\mathbf{i}+3\mathbf{j}+2\mathbf{k})\bullet(-\mathbf{i}+3\mathbf{j}+2\mathbf{k})=10\times(-1)+3\times3+2\times2=3$, which matches plane Π. Point B lies in the plane.

d The normal vector of the plane $\mathbf{r}\bullet(-\mathbf{i}+3\mathbf{j}+2\mathbf{k})=3$ is $\langle-1,3,2\rangle$.

The normal is perpendicular to the plane.

The line, L, has vector $\langle5,3,-2\rangle$.

$\langle-1,3,2\rangle\bullet\langle5,3,-2\rangle=-5+9-4=0$

The line is perpendicular to the normal of the plane and hence it lies in the plane.

Question 15

$$\begin{pmatrix}-10\\20\\-12\end{pmatrix}+t\begin{pmatrix}5\\-10\\6\end{pmatrix}=\begin{pmatrix}-3\\-8\\2\end{pmatrix}+t\begin{pmatrix}4\\-6\\4\end{pmatrix}$$

i.e. $-10+5t=-3+4t$, solving gives $t=7$

$20-10t=-8-6t$, solving gives $t=7$.

$-12+6t=2+4t$, solving gives $t=7$.

The particles will collide at $t = 7$ seconds at the point with position vector $\begin{pmatrix}-10\\20\\-12\end{pmatrix}+7\begin{pmatrix}5\\-10\\6\end{pmatrix}=\begin{pmatrix}25\\-50\\30\end{pmatrix}$.

Question 16

The given line is parallel to $\begin{pmatrix}1\\-3\\-5\end{pmatrix}$ so the plane must be parallel to $\begin{pmatrix}1\\-3\\-5\end{pmatrix}$.

For line $\mathbf{r}=\begin{pmatrix}2\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\-5\end{pmatrix}$, setting $\lambda=0$ gives $\begin{pmatrix}2\\2\\1\end{pmatrix}$ as a point on the line and hence in the plane.

Thus $\begin{pmatrix}2\\2\\1\end{pmatrix}-\begin{pmatrix}1\\2\\0\end{pmatrix}$, i.e. $\begin{pmatrix}1\\0\\1\end{pmatrix}$, must be a vector parallel to the plane.

The required equation can be written: $\mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}+\mu\begin{pmatrix}1\\-3\\-5\end{pmatrix}$ (can be written other ways).

From this:

$x=1+\lambda+\mu$

$y=2-3\mu$

$z=\lambda-5\mu$

Solving gives $x=-2y+z+5$

$x+2y-z=5$ in Cartesian form.

In the form $\mathbf{r}\bullet\mathbf{n}=\text{c}$

$$\mathbf{r}\bullet\begin{pmatrix}1\\2\\-1\end{pmatrix}=5$$

$$\left.\begin{cases}x=1+\lambda+\mu\\y=2-3\mu\\z=\lambda-5\mu\end{cases}\right|x,\lambda,\mu$$

$\{x=-2\cdot y+z+5,\ \lambda=-0.3333333333\cdot(5\cdot y-3\cdot z-10),$

Question 17

Line with vector equation $\mathbf{r} = 2\mathbf{i} + 13\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$ meets the plane $\mathbf{r} \bullet (2\mathbf{i} - \mathbf{j} - \mathbf{k}) = 11$ at a point with position vector $\mathbf{a}$.

$\mathbf{a} = 2\mathbf{i} + 13\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$ and $\mathbf{a} \bullet (2\mathbf{i} - \mathbf{j} - \mathbf{k}) = 11$

$$\begin{pmatrix} 2-\lambda \\ 13+3\lambda \\ 1-2\lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 11$$

$$4 - 2\lambda - 13 - 3\lambda - 1 + 2\lambda = 11$$

$$\lambda = -7$$

$$\begin{aligned} \mathbf{a} &= 2\mathbf{i} + 13\mathbf{j} + \mathbf{k} - 7(-\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \\ &= 9\mathbf{i} - 8\mathbf{j} + 15\mathbf{k} \end{aligned}$$

The position vector of the point where the line meets the plane is $9\mathbf{i} - 8\mathbf{j} + 15\mathbf{k}$.

Question 18

$$1200\mathbf{i} + 3000\mathbf{j} + 900\mathbf{k} + t(2000\mathbf{i} - 3600\mathbf{j} + 1000\mathbf{k}) = 5750\mathbf{i} - 13250\mathbf{j} + 3370\mathbf{k} + t(600\mathbf{i} + 1400\mathbf{j} + 240\mathbf{k})$$

$1200 + 2000t = 5750 + 600t$, $t = 3.25$ seconds

$3000 - 3600t = -13250 + 1400t$, $t = 3.25$ seconds

$900 + 1000t = 3370 + 240t$, $t = 3.25$ seconds

The spacecraft and space debris have the same position vector after 3.25 seconds.

They collide at the point with position vector $7700\mathbf{i} - 8700\mathbf{j} + 4150\mathbf{k}$.

Question 19

Position of B after 10 minutes is $150\mathbf{i} + 470\mathbf{j} + 2\mathbf{k} + \frac{10}{60}(300\mathbf{i} + 180\mathbf{j}) = 200\mathbf{i} + 500\mathbf{j} + 2\mathbf{k}$.

Position of A needs to be the same after 10 minutes.

$$80\mathbf{i} + 400\mathbf{j} + 3\mathbf{k} + \frac{10}{60}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 200\mathbf{i} + 500\mathbf{j} + 2\mathbf{k}$$

$$80 + \frac{10}{60}x = 200,\ x = 720$$

$$400 + \frac{10}{60}y = 500,\ y = 600$$

$$3 + \frac{10}{60}z = 2,\quad z = -6$$

The fighter pilot will maintain a velocity given by vector $(720\mathbf{i} + 600\mathbf{j} - 6\mathbf{k})$ km/h in order to intercept with the supply plane.

Question 20

a Plane $\prod_1$ is perpendicular to $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Plane $\prod_2$ is perpendicular to $\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$.

$\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} = -\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and as one is a scalar multiple of the other, it follows that $\prod_1 = \prod_2$.

b $d = \sqrt{15^2 - 12^2} = 9\,\text{units}$.

The planes are 9 units apart.

Question 21

$x_A = 30 + 5t \quad \text{and} \quad x_B = 2 + 8t$
$y_A = -37 + 8t \text{ and } \quad y_B = 40$
$z_A = -30 + 3t \text{ and } \quad z_B = 26 - 2t$

$$d = [2 + 8t - (30 + 5t), 40 - (-37 + 8t), 26 - 2t - (-30 + 3t)]$$
$$= [-28 + 3t, 77 - 8t, 56 - 5t]$$

$$|\overline{d}|^2 = (-28 + 3t)^2 + (77 - 8t)^2 + (56 - 5t)^2$$
$$= 98t^2 - 1960t + 9849 = 98\left(t^2 - 20t + \frac{201}{2}\right)$$
$$= 98\left[(t - 10)^2 - 10^2 + \frac{201}{2}\right] = 98(t - 10)^2 + 98\left(\frac{1}{2}\right)$$
$$= 98(t - 10)^2 + 49$$

The minimum separation distance between the particles is $\sqrt{49} = 7\,\text{m}$

The particles are this distance apart after 10 seconds.

Exercise 5D

Question 1

Given $|\mathbf{r}| = 16$. The sphere described has centre (0, 0, 0) and a radius of 16 units.

Question 2

Given $x^2 + y^2 + z^2 = 100$. The sphere described has centre (0, 0, 0) and a radius of 10 units.

Question 3

Given $|\mathbf{r}| = 25$. The sphere described has centre (1, 1, 1) and a radius of 25 units.

Question 4

Given $|\mathbf{r} - 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}| = 18$, so $|\mathbf{r} - (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})| = 18$. The sphere described has centre (2, –3, 4) and a radius of 18 units.

Question 5

Given $(x-3)^2 + (y+1)^2 + (z-2)^2 = 10$. The sphere described has centre (3, –1, 2) and a radius of $\sqrt{10}$ units.

Question 6

Given $(x+4)^2 + (y-1)^2 + z^2 = 25$. The sphere described has centre (–4, 1, 0) and a radius of 5 units.

Question 7

Given $x^2 + y^2 - 8y + 16 + z^2 = 50$.

Rearrange to get $x^2 + (y-4)^2 + z^2 = 50$.

The sphere described has centre (0, 4, 0) and a radius of $5\sqrt{2}$ units.

Question 8

Given $x^2 + y^2 + z^2 - 2x + 6y = 15$.

Rearrange to get $(x-1)^2 - 1 + (y+3)^2 - 9 + z^2 = 15$

$(x-1)^2 + (y+3)^2 + z^2 = 25$

The sphere described has centre (1, –3, 0) and a radius of 5 units.

Question 9

Given $x^2 + y^2 + z^2 - 6y + 2z = 111$.

Rearrange to get $x^2 + (y-3)^2 - 9 + (z+1)^2 - 1 = 111$

$x^2 + (y-3)^2 + (z+1)^2 = 121$

The sphere described has centre (0, 3, –1) and a radius of 11 units.

Question 10

Given $x^2 + y^2 + z^2 + 8x - 2y + 2z = 7$.

Rearrange to get $(x+4)^2 - 16 + (y-1)^2 - 1 + (z+1)^2 - 1 = 7$

$(x+4)^2 + (y-1)^2 + (z+1)^2 = 25$

The sphere described has centre (–4, 1, –1) and a radius of 5 units.

Question 11

Given $|\mathbf{r}| = 5$

$|(2, -3, 4)| = \sqrt{29} \approx 5.4$

The distance of the point (2, –3, 4) from the centre of the circle is greater than the radius, the point lies outside the circle.

Question 12

Given $|\mathbf{r}| = 7$

$|(-2, 3, 6)| = \sqrt{49} = 7$

The distance of the point (–2, 3, 6) from the centre of the circle is equal to the radius, the point lies on the circle.

Question 13

Given $|\mathbf{r}| = 16$

$|(7,12,9)| = \sqrt{274} \approx 16.6$

The distance of the point (7, 12, 9) from the centre of the circle is greater than the radius, the point lies outside the circle.

Question 14

Given $|\mathbf{r} - (\mathbf{i} + \mathbf{j} - \mathbf{k})| = 8$

Vector from centre of the circle to (3, 1, 0) is (2, 0, 1).

$|(2,0,1)| = \sqrt{5} \approx 2.2$

The distance of the point (3, 1, 0) from the centre of the circle is less than the radius, the point lies inside the circle.

Question 15

Given $|\mathbf{r} - (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})| = 5$

Vector from centre of circle to (3, 5, 2) is (1, 6, –1).

$|(1,6,-1)| = \sqrt{38} \approx 6.2$

The distance of the point (3, 5, 2) from the centre of the circle is greater than the radius, the point lies outside the circle.

Question 16

Given $|\mathbf{r} - (7\mathbf{i} + 10\mathbf{j} + 2\mathbf{k})| = 13$

Vector from the centre of the circle to (2, –2, 2) is (–5, –12, 0)

$|(-5,-12,0)| = \sqrt{169} = 13$

The distance of the point (2, –2, 2) from the centre of the circle is equal to the radius, the point lies on the circle.

Question 17

Given $(x-1)^2 + (y+3)^2 + (z-2)^2 = 36$.

The sphere described has centre (1, –3, 2) and a radius of 6 units.

The distance of (5, –6, –1) from the centre of the circle is $\sqrt{4^2 + 3^2 + 3^2} = \sqrt{34} \approx 5.8$

The distance of the point (5, –6, –1) from the centre of the circle is less than the radius, the point lies inside the circle.

Question 18

Given $x^2 + y^2 + z^2 - 4x - 3y - z = 61$.

Rearrange to get $(x-2)^2 + (y-1.5)^2 + (z-0.5)^2 = 67.5$

The sphere described has centre (2, 1.5, 0.5) and a radius of approximately 8.22 units.

The distance of (–1, 0, 8) from the centre of the circle is $\sqrt{3^2 + 1.5^2 + 7.5^2} = \sqrt{67.5} \approx 8.22$

The distance of the point (–1, 0, 8) from the centre of the circle is equal to the radius, the point lies on the circle.

Question 19

Circle centre (1, 1, –3) and radius $5\sqrt{2}$.

Distance from centre to point A is equal to the radius, use the equation $\sqrt{(a-1)^2 + 4^2 + 5^2} = 5\sqrt{2}$.

Solving gives $a = 4$ (as only positive values are valid as stated in the question).

Distance from centre to point B is equal to the radius, can use the equation
$\sqrt{(-5)^2 + (b-1)^2 + 0^2} = 5\sqrt{2}$.

Solving gives $b = 6$ (as only positive values are valid as stated in the question).

Distance from the centre to point C is equal to the radius, use the equation $\sqrt{1^2 + 0^2 + (c+3)^2} = 5\sqrt{2}$.

Solving gives $c = 4$ (as only positive values are valid as stated in the question).

Question 20

Given sphere $\left|\mathbf{r} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}\right| = 5\sqrt{2}$ and line $\mathbf{r} = \begin{pmatrix} -2 \\ 16 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix}$.

Find where the line cuts the sphere by substitution:

$$\left|\begin{pmatrix} -2 \\ 16 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}\right| = \left|\begin{pmatrix} -3 \\ 17 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix}\right| = 5\sqrt{2}$$

$$(-3-2\lambda)^2 + (17+5\lambda)^2 + (-4-2\lambda)^2 = 50$$

Solving gives $\lambda = -4, -2$

So the position vector of the points of intersection are:

when $\lambda = -4$, $\mathbf{r} = \begin{pmatrix} -2 \\ 16 \\ -1 \end{pmatrix} - 4 \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 7 \end{pmatrix}$

when $\lambda = -2$, $\mathbf{r} = \begin{pmatrix} -2 \\ 16 \\ -1 \end{pmatrix} - 2 \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$

Question 21

Given sphere $|\mathbf{r}-(4\mathbf{i}+\mathbf{j}+3\mathbf{k})|=7$ and line $\mathbf{r}=14\mathbf{i}-9\mathbf{k}+\lambda(4\mathbf{i}+\mathbf{j}-9\mathbf{k})$.

Find where the line cuts the sphere by substitution:

$$|14\mathbf{i}-9\mathbf{k}+\lambda(4\mathbf{i}+\mathbf{j}-9\mathbf{k})-(4\mathbf{i}+\mathbf{j}+3\mathbf{k})|=7$$

$$|(10+4\lambda)\mathbf{i}+(-1+\lambda)\mathbf{j}+(-12-9\lambda)\mathbf{k}|=7$$

$$(10+4\lambda)^2+(\lambda-1)^2+(-12-9\lambda)^2=49$$

$\lambda=-1,-2$

So the position vector of the points of intersection are:

When $\lambda=-1$, $\mathbf{r}=14\mathbf{i}-9\mathbf{k}-1(4\mathbf{i}+\mathbf{j}-9\mathbf{k})$

$\mathbf{r}=10\mathbf{i}-\mathbf{j}$

When $\lambda=-2$, $\mathbf{r}=14\mathbf{i}-9\mathbf{k}-2(4\mathbf{i}+\mathbf{j}-9\mathbf{k})$

$\mathbf{r}=6\mathbf{i}-2\mathbf{j}+9\mathbf{k}$

Question 22

Given sphere $|\mathbf{r}-(3\mathbf{i}-\mathbf{j}+4\mathbf{k})|=5$ and line $\mathbf{r}=-2\mathbf{i}-\mathbf{j}-11\mathbf{k}+\lambda(3\mathbf{i}+4\mathbf{k})$.

Find where the line cuts the sphere by substitution:

$$|-2\mathbf{i}-\mathbf{j}-11\mathbf{k}+\lambda(3\mathbf{i}+4\mathbf{k})-(3\mathbf{i}-\mathbf{j}+4\mathbf{k})|=5$$

$$|-5\mathbf{i}-15\mathbf{k}+\lambda(3\mathbf{i}+4\mathbf{k})|=5$$

$$(-5+3\lambda)^2+(-15+4\lambda)^2=25$$

Solving gives $\lambda=3$

As there is only one value for λ, there is only one point where the line touches the sphere.

So the position vector of the point of intersection is:

When $\lambda=3$, $\mathbf{r}=-2\mathbf{i}-\mathbf{j}-11\mathbf{k}+3(3\mathbf{i}+4\mathbf{k})$

$\mathbf{r}=7\mathbf{i}-\mathbf{j}+\mathbf{k}$

Question 23

Given sphere $\left|\mathbf{r}-\begin{pmatrix}-2\\1\\3\end{pmatrix}\right|=7$ and line $\mathbf{r}=\begin{pmatrix}9\\18\\20\end{pmatrix}+\lambda\begin{pmatrix}-1\\-4\\-3\end{pmatrix}$.

Find where the line cuts the sphere by substitution:

$$\left|\begin{pmatrix}9\\18\\20\end{pmatrix}+\lambda\begin{pmatrix}-1\\-4\\-3\end{pmatrix}-\begin{pmatrix}-2\\1\\3\end{pmatrix}\right|=\left|\begin{pmatrix}11\\17\\17\end{pmatrix}+\lambda\begin{pmatrix}-1\\-4\\-3\end{pmatrix}\right|=7$$

$$(11-\lambda)^2+(17-4\lambda)^2+(17-3\lambda)^2=49$$

Solving gives $\lambda=5$.

As there is only one value for λ, there is only one point where the line touches the sphere.

So the position vector of the point of intersection is:

When $\lambda=5$, $\mathbf{r}=\begin{pmatrix}9\\18\\20\end{pmatrix}+5\begin{pmatrix}-1\\-4\\-3\end{pmatrix}=\begin{pmatrix}4\\-2\\5\end{pmatrix}$

Exercise 5E

Question 1

Scalar product approach

$$\overrightarrow{BC} = -\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

$$\overrightarrow{AD} = -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1+\lambda \\ -2+2\lambda \\ 1-3\lambda \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 1+\lambda \\ -2+2\lambda \\ 1-3\lambda \end{pmatrix} = 0$$

$$1+\lambda-4+4\lambda-3+9\lambda = 0$$

$$14\lambda - 6 = 0$$

$$\lambda = \frac{3}{7}$$

$$\overrightarrow{AD} = \begin{pmatrix} \frac{10}{7} \\ -\frac{8}{7} \\ -\frac{2}{7} \end{pmatrix}$$

Hence $d = \sqrt{\frac{24}{7}} = \frac{\sqrt{168}}{7} = \frac{2\sqrt{42}}{7}$ units of length.

Vector product approach.

$$\begin{aligned} \text{Required distance } &= \left|\overrightarrow{CA}\right| \sin\theta \\ &= \left(\left|\overrightarrow{BC}\right|\left|\overrightarrow{CA}\right|\sin\theta\right) \div \left|\overrightarrow{BC}\right| \\ &= \left|\overrightarrow{BC} \times \overrightarrow{CA}\right| \div \left|\overrightarrow{BC}\right| \end{aligned}$$

Now $\overrightarrow{BC} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

and $\overrightarrow{CA} = -2\mathbf{i} + 2\mathbf{k}$

Thus $\left|\overrightarrow{BC} \times \overrightarrow{CA}\right| = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$

$$\left|\overrightarrow{BC} \times \overrightarrow{CA}\right| \div \left|\overrightarrow{BC}\right| = 4\sqrt{3} \div \sqrt{14} = \frac{2\sqrt{42}}{7}$$

[1,2,−3]

$[1 \;\; 2 \;\; -3]$

[−2,0,2]

$[-2 \;\; 0 \;\; 2]$

crossP([1 2 −3],[−2 0 2])

$[4 \;\; 4 \;\; 4]$

$\sqrt{4^2+4^2+4^2}$

$4\cdot\sqrt{3}$

$\sqrt{1^2+2^2+(-3)^2}$

$\sqrt{14}$

$4\cdot\sqrt{3}/\sqrt{14}$

$\frac{2\cdot\sqrt{42}}{7}$

Question 2

Scalar product approach

$$\overrightarrow{BC} = -\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$\overrightarrow{AD} = -\begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2+\lambda \\ 3-2\lambda \\ -6+5\lambda \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -2+\lambda \\ 3-2\lambda \\ -6+5\lambda \end{pmatrix} = 0$$

$$\begin{aligned} -2+\lambda-6+4\lambda-30+25\lambda &= 0 \\ 30\lambda-38 &= 0 \\ \lambda &= \frac{19}{15} \end{aligned}$$

$$\overrightarrow{AD} = \begin{pmatrix} -\frac{11}{15} \\ \frac{7}{15} \\ \frac{1}{3} \end{pmatrix}$$

Hence $d = \dfrac{\sqrt{195}}{15}$ units of length.

Vector product approach.

$$\begin{aligned} \text{Required distance } &= \left|\overrightarrow{CA}\right|\sin\theta \\ &= \left(\left|\overrightarrow{BC}\right|\left|\overrightarrow{CA}\right|\sin\theta\right) \div \left|\overrightarrow{BC}\right| \\ &= \left|\overrightarrow{BC}\times\overrightarrow{CA}\right| \div \left|\overrightarrow{BC}\right| \end{aligned}$$

Now $\overrightarrow{BC} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

and $\overrightarrow{CA} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

Thus $\left|\overrightarrow{BC}\times\overrightarrow{CA}\right| = 3\mathbf{i} + 4\mathbf{j} + 1\mathbf{k}$

$$\left|\overrightarrow{BC}\times\overrightarrow{CA}\right| \div \left|\overrightarrow{BC}\right| = \sqrt{26} \div \sqrt{30} = \frac{\sqrt{195}}{15}$$

```
[1,-2,5]
                                   [1 -2 5]
[1,-1,1]
                                   [1 -1 1]
crossP([1 -2 5],[1 -1 1])
                                    [3 4 1]
√(3²+4²+1²)
                                        √26
√(1²+(-2)²+5²)
                                        √30
√26/√30
                                     √195
                                     ----
                                      15
```

Question 3

Scalar product approach

$$\overrightarrow{BC} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

$$\overrightarrow{AD} = \begin{pmatrix} 2+\lambda \\ -3\lambda \\ 3+4\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+\lambda \\ -2-3\lambda \\ 2+4\lambda \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1+\lambda \\ -2-3\lambda \\ 2+4\lambda \end{pmatrix} = 0$$

$$1+\lambda+6+9\lambda+8+16\lambda = 0$$

$$26\lambda+15 = 0$$

$$\lambda = \frac{-15}{26}$$

$$\overrightarrow{AD} = \begin{pmatrix} \frac{11}{26} \\ -\frac{7}{26} \\ -\frac{4}{13} \end{pmatrix}$$

Hence $d = \dfrac{3\sqrt{26}}{26}$ units of length.

Vector product approach

From the given information, the position vector of point A is $\mathbf{i}+2\mathbf{j}+\mathbf{k}$, the position vector of point B is $2\mathbf{i}+3\mathbf{k}$ and the position vector of point C is $3\mathbf{i}-3\mathbf{j}+7\mathbf{k}$.

$$\text{Required distance } = \left|\overrightarrow{CA}\right|\sin\theta$$

$$= \left(\left|\overrightarrow{BC}\right|\left|\overrightarrow{CA}\right|\sin\theta\right) \div \left|\overrightarrow{BC}\right|$$

$$= \left|\overrightarrow{BC}\times\overrightarrow{CA}\right| \div \left|\overrightarrow{BC}\right|$$

Now $\overrightarrow{BC} = \mathbf{i}-3\mathbf{j}+4\mathbf{k}$

and $\overrightarrow{CA} = -2\mathbf{i}+5\mathbf{j}-6\mathbf{k}$

Thus $\overrightarrow{BC}\times\overrightarrow{CA} = -2\mathbf{i}-2\mathbf{j}-\mathbf{k}$

$$\left|\overrightarrow{BC}\times\overrightarrow{CA}\right| \div \left|\overrightarrow{BC}\right| = 3 \div \sqrt{26} = \frac{3\sqrt{26}}{26}$$

```
[1,-3,4]
                                   [1 -3 4]
[-2,5,-6]
                                   [-2 5 -6]
crossP([1 -3 4],[-2 5 -6])
                                   [-2 -2 -1]
√((-2)^2+(-2)^2+(-1)^2)
                                           3
√(1^2+3^2+4^2)
                                         √26
3/√26
                                     3·√26/26
```

Miscellaneous Exercise 5

Question 1

a $f(3) = 3 \times 3 - 2 = 7$

b $f(-3) = 3(-3) - 2 = -11$

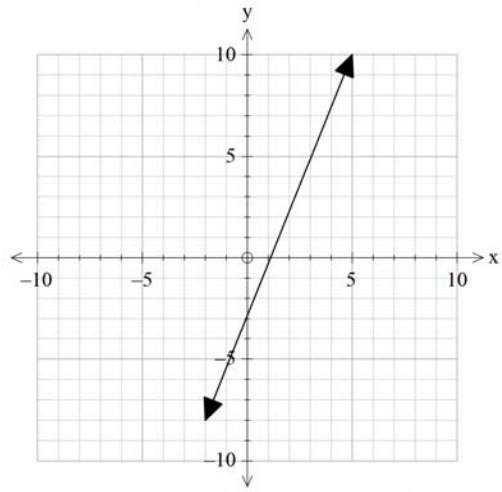
c $g(3) = f(3) = 7$

d $g(-3) = f(3) = 7$

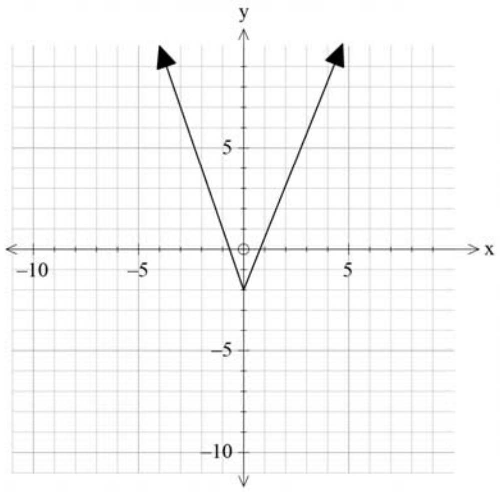
e $f(5) = 3 \times 5 - 2 = 13$

f $g(-5) = f(5) = 13$

g Graph of $y = f(x)$, i.e. $y = 3x - 2$



Graph of $y = f(|x|)$, i.e. $y = 3|x| - 2$



Question 2

a $\left(\dfrac{1+9}{2}, \dfrac{2-4}{2}\right) = (5, -1)$

b $d = \sqrt{4^2 + 3^2} = \sqrt{25} = 5\,\text{units}$

$r = 5$ units

c $|\mathbf{r} - (5\mathbf{i} - \mathbf{j})| = 5$

Question 3

a Centre (3, –2) and radius = 7 units.

b $|\mathbf{r}-(2\mathbf{i}+7\mathbf{j})|=11$

Centre (2, 7) and radius = 11 units.

c Centre (3, –2) and radius = 4 units.

d Centre (–1, –7) and radius = $2\sqrt{5}$ units.

e
$$x^2+y^2-8x=4y+5$$
$$(x-4)^2-16+(y-2)^2-4=5$$
$$(x-4)^2+(y-2)^2=25$$

Centre (4, 2) and radius = 5 units.

f
$$x^2+6x+y^2-14y=42$$
$$(x+3)^2-9+(y-7)^2-49=42$$
$$(x+3)^2+(y-7)^2=100$$

Centre (–3, 7) and radius = 10 units.

Question 4

Given line L1 with equation $\mathbf{r}=2\mathbf{i}+\mathbf{j}+\lambda(3\mathbf{i}-\mathbf{j})$ and L2 with equation $\mathbf{r}=3\mathbf{i}-\mathbf{j}+\mu(2\mathbf{i}+3\mathbf{j})$.

$$\cos\theta=\frac{\begin{pmatrix}3\\-1\\0\end{pmatrix}\bullet\begin{pmatrix}2\\3\\0\end{pmatrix}}{\left|\begin{pmatrix}3\\-1\\0\end{pmatrix}\right|\left|\begin{pmatrix}2\\3\\0\end{pmatrix}\right|}=\frac{3}{\sqrt{10}\sqrt{13}}\approx 0.2631$$

$\theta\approx 74.74\approx 75°$

Question 5

a Range: $\{y \in \mathbb{R} : y \geq 0\}$

b Range: $\{y \in \mathbb{R} : y \geq 3\}$

c Range: $\{y \in \mathbb{R} : y \geq 0\}$

d Range: $\{y \in \mathbb{R} : y \geq 0\}$

e Range: $\{y \in \mathbb{R} : y \geq 3\}$

f Range: $\{y \in \mathbb{R} : y \geq 0\}$

Question 6

$$x^2 + 2x + y^2 - 10y + a = 0$$
$$(x+1)^2 - 1 + (y-5)^2 - 25 = -a$$
$$(x+1)^2 + (y-5)^2 = 26 - a$$

$a < 26$ in order for the radius of the circle to be greater than zero.

Question 7

$f \circ g\,(x) = \dfrac{3}{2x-1}$, domain $\left\{x \in \mathbb{R} : x \neq \dfrac{1}{2}\right\}$, range $\{y \in \mathbb{R} : y \neq 0\}$

$g \circ f\,(x) = 2\left(\dfrac{3}{x}\right) - 1$, domain $\{x \in \mathbb{R} : x \neq 0\}$, range $\{y \in \mathbb{R} : y \neq -1\}$

Question 8

$f \circ g\,(x) = \sqrt{x^2 + 4}$, domain $\{x \in \mathbb{R}\}$, range $\{y \in \mathbb{R} : y \geq 2\}$

$g \circ f\,(x) = x + 4$, domain $\{x \in \mathbb{R} : x \geq -3\}$, range $\{y \in \mathbb{R} : y \geq 1\}$

Question 9

a $\overline{z} = 8\,\text{cis}\left(-\frac{\pi}{6}\right)$

b
$$z + \overline{z} = 8\,\text{cis}\left(\frac{\pi}{6}\right) + 8\,\text{cis}\left(-\frac{\pi}{6}\right)$$
$$= 8\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) + 8\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$$
$$= 8 \times \frac{\sqrt{3}}{2} + 8 \times \frac{\sqrt{3}}{2}$$
$$= 8\sqrt{3}\ \text{cis}\ 0$$

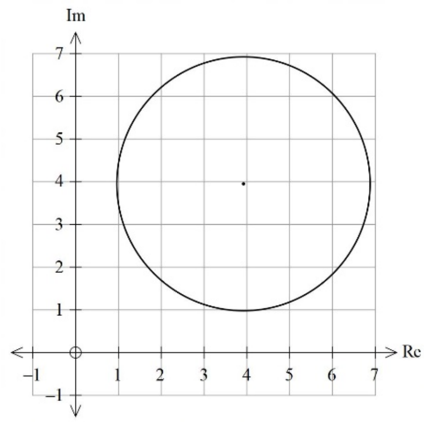
c
$$z - \overline{z} = 8\text{cis}\left(\frac{\pi}{6}\right) - 8\text{cis}\left(-\frac{\pi}{6}\right)$$
$$= 8\cos\frac{\pi}{6} + 8i\sin\frac{\pi}{6} - 8\cos-\frac{\pi}{6} + 8i\sin-\frac{\pi}{6}$$
$$= 8 \times 1$$
$$= 8\,\text{cis}\frac{\pi}{2}$$

d
$$z\overline{z} = 8\,\text{cis}\left(\frac{\pi}{6}\right) \times 8\,\text{cis}\left(-\frac{\pi}{6}\right)$$
$$= 64\,\text{cis}\left(\frac{\pi}{6} - \frac{\pi}{6}\right)$$
$$= 64\,\text{cis}\ 0$$

e
$$\frac{z}{\overline{z}} = \frac{8\,\text{cis}\left(\frac{\pi}{6}\right)}{8\,\text{cis}\left(-\frac{\pi}{6}\right)} = \text{cis}\left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = \text{cis}\left(\frac{\pi}{3}\right)$$

Question 10

For $\{z : |z - (4 + 4i)| = 3\}$



a The minimum possible value of $\text{Im}(z)$ is 1.

b The maximum possible value of $\text{Re}(z)$ is 7.

c The minimum possible value of
$|z| = \sqrt{4^2 + 4^2} - 3 = 4\sqrt{2} - 3$

d The maximum possible value of $|z| = 4\sqrt{2} + 3$

e The maximum possible value of $|z| = 4\sqrt{2} + 3$

Question 11

Given $z = 3\operatorname{cis}\frac{5\pi}{6}$ and $w = 2\operatorname{cis}\left(-\frac{2\pi}{3}\right)$.

a $2z = 6\operatorname{cis}\frac{5\pi}{6}$

b $3w = 6\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

c $zw = 3\operatorname{cis}\frac{5\pi}{6} \times 2\operatorname{cis}\left(-\frac{2\pi}{3}\right) = 6\operatorname{cis}\frac{\pi}{6}$

d

$$\frac{z}{w} = \frac{3\operatorname{cis}\frac{5\pi}{6}}{2\operatorname{cis}\left(-\frac{2\pi}{3}\right)} = \frac{3\operatorname{cis}\left(\frac{5\pi}{6}+\frac{2\pi}{3}\right)}{2}$$

$$= \frac{3}{2}\operatorname{cis}\frac{9\pi}{6} = \frac{3}{2}\operatorname{cis}\frac{3\pi}{2} = 1.5\operatorname{cis}\left(-\frac{\pi}{2}\right)$$

e $iz = 1\operatorname{cis}\frac{\pi}{2} \times 3\operatorname{cis}\frac{5\pi}{6} = 3\operatorname{cis}\frac{8\pi}{6} = 3\operatorname{cis}\left(-\frac{4\pi}{6}\right) = 3\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

f $-w = \operatorname{cis}\pi \times 2\operatorname{cis}\left(-\frac{2\pi}{3}\right) = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$

g $\overline{z} = 3\operatorname{cis}\left(-\frac{5\pi}{6}\right)$

h $zw = 6\operatorname{cis}\frac{\pi}{6}$, $\overline{zw} = 6\operatorname{cis}\left(-\frac{\pi}{6}\right)$

i $\overline{z}\,\overline{w} = 3\operatorname{cis}\left(-\frac{5\pi}{6}\right) \times 2\operatorname{cis}\left(\frac{2\pi}{3}\right) = 6\operatorname{cis}\left(-\frac{\pi}{6}\right)$

j $z^2w^3 = 3^2\operatorname{cis}\left(2\times\frac{5\pi}{6}\right) \times 2^3\operatorname{cis}\left[3\times\left(-\frac{2\pi}{3}\right)\right] = 72\operatorname{cis}\left(-\frac{\pi}{3}\right)$

Question 12

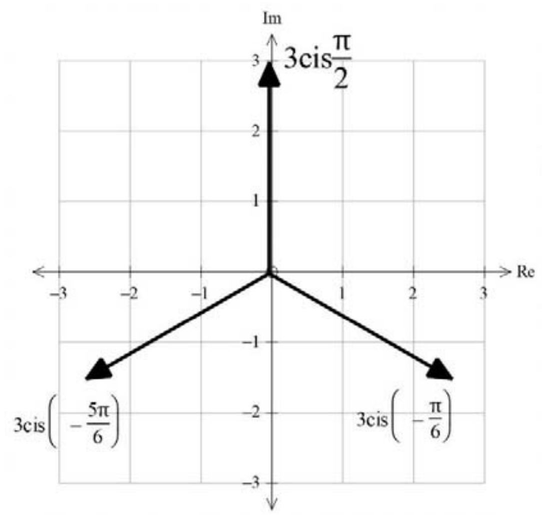
$$z^3 = \frac{27}{i} = \frac{27}{\text{cis}\frac{\pi}{2}} = 27\text{cis}\left(-\frac{\pi}{2}\right)$$

$$z = \sqrt[3]{27}\text{cis}\left(\frac{\pi}{2}\right)$$

$$= 3\,\text{cis}\left(\frac{\pi}{2}\right),\ 3\,\text{cis}\left(\frac{\pi}{2}+\frac{2\pi}{3}\right),\ 3\,\text{cis}\left(\frac{\pi}{2}+\frac{4\pi}{3}\right)$$

$$= 3\,\text{cis}\left(\frac{\pi}{2}\right),\ 3\,\text{cis}\left(\frac{7\pi}{6}\right),\ 3\,\text{cis}\left(\frac{11\pi}{6}\right)$$

$$= 3\,\text{cis}\left(\frac{\pi}{2}\right),\ 3\,\text{cis}\left(-\frac{5\pi}{6}\right),\ 3\,\text{cis}\left(-\frac{\pi}{6}\right)$$



Question 13

$$z+\frac{1}{z} = \left(-1+\sqrt{3}\,i\right)+\frac{1}{-1-\sqrt{3}\,i} = \frac{\left(-1+\sqrt{3}\,i\right)\left(-1-\sqrt{3}\,i\right)}{-1-\sqrt{3}\,i}+\frac{1}{-1-\sqrt{3}\,i}$$

$$= \frac{1-3i^2+1}{-1-\sqrt{3}\,i} = \frac{5}{-1-\sqrt{3}\,i}\times\frac{-1+\sqrt{3}\,i}{-1+\sqrt{3}\,i} = \frac{-5+5\sqrt{3}\,i}{4}$$

$$z = \frac{5}{2}\text{cis}\left(-\frac{\pi}{3}\right)$$

$$z^4 = \frac{5^4}{2^4}\text{cis}\left(-\frac{4\pi}{3}\right) = \frac{5^4}{2^4}\text{cis}\left(\frac{2\pi}{3}\right)$$

$$z-\frac{1}{\overline{z}} = \left(-1+\sqrt{3}\,i\right)-\frac{1}{-1-\sqrt{3}\,i}$$

$$= \frac{\left(-1+\sqrt{3}\,i\right)\left(-1-\sqrt{3}\,i\right)}{-1-\sqrt{3}\,i}-\frac{1}{-1-\sqrt{3}\,i}$$

$$= \frac{1-3i^2-1}{-1-\sqrt{3}\,i} = \frac{3}{-1-\sqrt{3}\,i}\times\frac{-1+\sqrt{3}\,i}{-1+\sqrt{3}\,i} = \frac{-3+3\sqrt{3}\,i}{4}$$

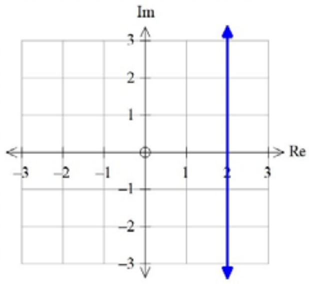
$$z = \frac{3}{2}\text{cis}\left(-\frac{\pi}{3}\right)$$

$$z^4 = \frac{3^4}{2^4}\text{cis}\left(-\frac{4\pi}{3}\right) = \frac{3^4}{2^4}\text{cis}\left(\frac{2\pi}{3}\right)$$

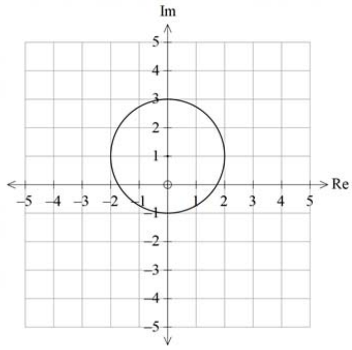
Question 14

a $z+\overline{z}=4$

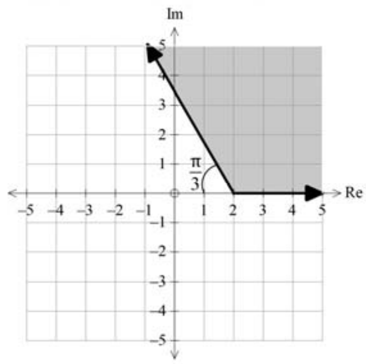
$$\begin{aligned} x+yi+x-yi&=4 \\ 2x&=4 \\ x&=2 \end{aligned}$$



b $|z-i|=2$



c $0 \leq \arg(z-2) \leq \dfrac{2\pi}{3}$



Question 15

Point A has position vector $\mathbf{i}+6\mathbf{j}$

Point B has position vector $-3\mathbf{i}+18\mathbf{j}+4\mathbf{k}$

$\overrightarrow{\mathrm{BA}}=4\mathbf{i}-12\mathbf{j}-4\mathbf{k}$

$\overrightarrow{\mathrm{AC}}=-\mathbf{i}+3\mathbf{j}+\mathbf{k}$

Point C has position vector $9\mathbf{j}+\mathbf{k}$

Question 16

a The two lines are parallel as $(3\mathbf{i}-2\mathbf{j}+\mathbf{k})\times 2=(6\mathbf{i}-4\mathbf{j}+2\mathbf{k})$

b $2-\lambda=-5+2\mu$

$3+3\lambda=3+\mu \Rightarrow \mu=3\lambda$

$10+\lambda=-5+2\mu$

$\lambda=1, \mu=3$ works for the first two equations but not the last.

These lines are not parallel or intersecting so they are skew lines.

c $-1-\lambda=\mu$

$5+\lambda=2 \Rightarrow \lambda=-3$

$\lambda=-7+2\mu$

$\lambda=-3, \mu=2.$

The lines intersect.

d The two lines are parallel as $(\mathbf{i}+\mathbf{j}-\mathbf{k})\times(-1)=(-\mathbf{i}-\mathbf{j}+\mathbf{k})$.

Question 17

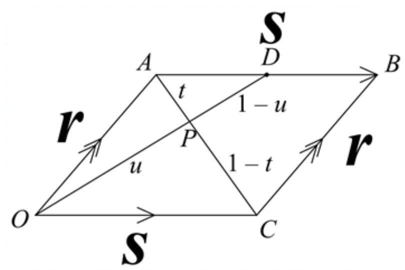
$1150+10t=1345-5t,\ t=13$ seconds

$827-2t=970-13t,\quad t=13$ seconds

When t = 13 seconds, the submarine is $-4\times 13=52$ metres below sea level.

Question 18

In the diagram, D is the midpoint of $\overline{AB}$, P is the point where $\overline{OD}$ intersects $\overline{AC}$.



$\overrightarrow{OA}=\mathbf{r}$, $\quad \overrightarrow{OC}=\mathbf{s}$, $\quad \overrightarrow{AC}=-\mathbf{r}+\mathbf{s}$

$\overrightarrow{AP}=t(-\mathbf{r}+\mathbf{s})$, $\quad \overrightarrow{OD}=\mathbf{r}+\frac{1}{2}\mathbf{s}$

$\overrightarrow{DP}=(1-u)\left(-\mathbf{r}-\frac{1}{2}\mathbf{s}\right)$

$\overrightarrow{AP}=-\mathbf{r}+u\left(\mathbf{r}+\frac{1}{2}\mathbf{s}\right)=(u-1)\mathbf{r}+\frac{u}{2}\mathbf{s}$

Comparing the coefficients of $\mathbf{r}$ and $\mathbf{s}$ in the two equations for $\overrightarrow{AP}$ gives:

$-t=u-1$

$t=\frac{u}{2} \Rightarrow u=2t$

Substitute $u=2t$ into $-t=u-1$

$-t=2t-1 \Rightarrow 3t=1$

So $t=\frac{1}{3}$

Which means that the line from O to the midpoint of $\overline{AB}$ intersects $\overline{AC}$ at the point of trisection of $\overline{AC}$ that is nearer to A.

Question 19

$$\mathbf{r}_{\mathrm{A}}(t)=\begin{pmatrix}600\\0\\0\end{pmatrix}+t\begin{pmatrix}-196\\213\\18\end{pmatrix},\qquad \mathbf{r}_{\mathrm{B}}(t)=\begin{pmatrix}2200\\4000\\600\end{pmatrix}+t\begin{pmatrix}-240\\100\\0\end{pmatrix}$$

$$\overrightarrow{\mathrm{AB}}=\begin{pmatrix}2200\\4000\\600\end{pmatrix}-\begin{pmatrix}600\\0\\0\end{pmatrix}=\begin{pmatrix}1600\\4000\\600\end{pmatrix}$$

$$\mathbf{v}_{\mathrm{A}}-\mathbf{v}_{\mathrm{B}}=\begin{pmatrix}44\\113\\18\end{pmatrix}$$

For the missiles to collide, $\overrightarrow{\mathrm{AB}}=t\begin{pmatrix}44\\113\\18\end{pmatrix}$

For the $\mathbf{i}$ components:

$$\begin{aligned}600-196t&=2200-240t\\44t&=1600\\t&=\frac{400}{11}\end{aligned}$$

For the $\mathbf{j}$ components:

$$\begin{aligned}213t&=4000+100t\\113t&=4000\\t&=\frac{4000}{113}\end{aligned}$$

For the $\mathbf{k}$ components:

$$\begin{aligned}18t&=600\\t&=33\frac{1}{3}\end{aligned}$$

As the t values are not the same for the $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components, so missile A does not intercept missile B.

To find how much the missile misses by (the minimum distance), let P be the point of closest approach.

$$\overrightarrow{\mathrm{BP}}=\overrightarrow{\mathrm{BA}}+\overrightarrow{\mathrm{BP}}=-\overrightarrow{\mathrm{AB}}+t\begin{pmatrix}44\\113\\18\end{pmatrix}=-\begin{pmatrix}1600\\4000\\600\end{pmatrix}+t\begin{pmatrix}44\\113\\18\end{pmatrix}$$

$$\overrightarrow{BP} \bullet \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix} = \left(-\begin{pmatrix} 1600 \\ 4000 \\ 600 \end{pmatrix} + t\begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix} \right) \bullet \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix} = 0$$

$$44(-1600) + 113(-4000) + 18(-600) + t(44 \times 44 + 113 \times 113 + 18 \times 18) = 0$$

$$t = \frac{533\,200}{15\,029} \approx 35.48 \text{ seconds}$$

$$\overrightarrow{BP} = -\begin{pmatrix} 1600 \\ 4000 \\ 18 \end{pmatrix} + 35.478\begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix} = \begin{pmatrix} -38.965 \\ 9.023 \\ 38.605 \end{pmatrix}$$

$$\left|\overrightarrow{BP}\right| = \sqrt{(-38.965)^2 + 9.023^2 + 38.065^2} = 55.59 \text{ m}$$

Missile A misses missile B by approximately 56 m.

After 20 seconds the position of missile A is $\begin{pmatrix} 600 \\ 0 \\ 0 \end{pmatrix} + 20\begin{pmatrix} -196 \\ 213 \\ 18 \end{pmatrix} = \begin{pmatrix} -3320 \\ 4260 \\ 360 \end{pmatrix}$

After 20 seconds the position of missile B is $\begin{pmatrix} 2200 \\ 4000 \\ 600 \end{pmatrix} + 20\begin{pmatrix} -240 \\ 100 \\ 0 \end{pmatrix} = \begin{pmatrix} -2600 \\ 6000 \\ 600 \end{pmatrix}$

After 20 seconds, $\overrightarrow{AB} = \begin{pmatrix} 720 \\ 1740 \\ 240 \end{pmatrix}$

$$15(\mathbf{v}_A - \mathbf{v}_B) = \overrightarrow{AB}$$

The velocity vector or missile B is known, to find the velocity vector of missile A:

$$15\mathbf{v}_A = \begin{pmatrix} 720 \\ 1740 \\ 240 \end{pmatrix} + 15\begin{pmatrix} -240 \\ 100 \\ 0 \end{pmatrix}$$

$$\mathbf{v}_A = \frac{1}{15}\begin{pmatrix} -2880 \\ 3240 \\ 240 \end{pmatrix} = \begin{pmatrix} -192 \\ 216 \\ 16 \end{pmatrix} \text{m/second}$$

The constant velocity that A must maintain during the final 15 seconds is $(-192\mathbf{i} + 216\mathbf{j} + 16\mathbf{k})$ m/s .